

An Inverse Uniqueness in Interior Transmission Problem and Its Eigenvalue Tunneling in Penetrable Simple Domains

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Abstract

We study an inverse uniqueness with a knowledge of spectral data in the interior transmission problem defined by an index of refraction in a simple domain. We expand the solution in such a domain into a series of one dimensional problems. For each one dimensional problem, we apply a value distribution theory in complex analysis to describe the eigenvalues of the system. By the orthogonality of the one dimensional system, we consider the uniqueness on the perturbation along each given incident angle.

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1 Introduction and Preliminaries

In this paper, we study the inverse spectral problem in the following homogeneous interior transmission problem

$$\begin{cases} \Delta w + k^2 n(x)w = 0, & \text{in } D; \\ \Delta v + k^2 v = 0, & \text{in } D; \\ w = v, & \text{on } \partial D; \\ \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu}, & \text{on } \partial D, \end{cases} \quad (1.1)$$

where ν is the unit outer normal; D is a simple domain in \mathbb{R}^3 containing the origin with \mathcal{C}^2 -boundary ∂D ; $n(0) = 1$, $n(x) \in \mathcal{C}^2(\mathbb{R}^3)$; $n(x) > 0$, for $x \in D$; $n(x) = 1$, for $x \notin D$. The equation (1.1) is called the homogeneous interior transmission eigenvalue problem. We say $k \in \mathbb{C}$ is an interior transmission eigenvalue of (1.1) if there is a nontrivial pair of solution (w, v) such that $w, v \in L^2(D)$, $w - v \in H_0^2(D)$.

The problem (1.1) occurs naturally when one considers the scattering of the plane waves by certain inhomogeneity inside the domain D , defined by an index of refraction in many models. The interior transmission eigenvalues play a role in the inverse scattering theory both in numerical computation and in theoretical scattering theory. For the origin of interior transmission eigenvalue problem, we refer to Colton and Monk [15], and Kirsch [20]. For theoretical study and historic literature, we refer to [16, 17, 27, 31]. The one-to-one correspondence between the radiating solution of the Helmholtz equation and its far field patterns is well-known, but the denseness of the far field patterns remains further research topics. It is also another subject of research interest to study the existence or location of the eigenvalues [4, 5, 11, 15, 17, 21, 23, 24, 28]. It is expected to find a Weyl's type of asymptotics for the interior transmission eigenvalues. In that case, the distribution of the eigenvalues is directly connected to certain invariant characteristics on the scatterer. In this regard, we apply the methods from entire function theory [9, 10, 22, 25, 26] to study the distributional laws of the eigenvalues. We also refer to [32] for the reconstruction of the interior transmission eigenvalues, and [33] for a numerical description on the distribution of the eigenvalues. For the non-symmetrically-stratified medium, there are not too many known results [5, 16]. In this paper, we mainly follow the methods in [1, 2, 11, 12, 13]

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to study the non-symmetrical scatterers as a series of one-dimensional problems along the rays starting from the origin. The analysis along each ray possibly has multiple intersection points with ∂D , so we expect certain tunneling effect in a penetrable domain. In this paper, the new perspective is the following asymptotic analysis inside and outside the perturbation.

We apply Rellich's expansion in scattering theory. Firstly we expand the solution (v, w) of (1.1) in two series of spherical harmonics by Rellich's lemma [17, p. 32,]:

$$v(x; k) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} a_{l,m} j_l(kr) Y_l^m(\hat{x}); \quad (1.2)$$

$$w(x; k) = \frac{1}{r} \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} b_{l,m} y_l(r) Y_l^m(\hat{x}), \quad (1.3)$$

where $r := |x|$, $0 \leq r < \infty$; $\hat{x} = (\theta, \varphi) \in \mathbb{S}^2$; j_l is the spherical Bessel function of first kind of order l . The summations converge uniformly and absolutely on suitable compact subsets in $|r| \geq R_0$, with some sufficiently large $R_0 > 0$. We note that the expansion (1.2) and (1.3) hold for Helmholtz equation without radial symmetry assumption on the perturbation [17, p. 31, Lemma 2.11]. The uniqueness of the expansions outside D are given by Rellich's lemma [17, p. 32]. Particularly, the spherical harmonics

$$Y_l^m(\theta, \varphi) := \sqrt{\frac{2l+1}{4\pi} \frac{(l-|m|)!}{(l+|m|)!}} P_l^{|m|}(\cos \theta) e^{im\varphi}, \quad m = -l, \dots, l; \quad l = 0, 1, 2, \dots, \quad (1.4)$$

form a complete orthonormal system in $\mathcal{L}^2(\mathbb{S}^2)$. Here,

$$P_n^m(t) := (1-t^2)^{m/2} \frac{d^m P_n(t)}{dt^m}, \quad m = 0, 1, \dots, n, \quad (1.5)$$

where the Legendre polynomials P_n , $n = 0, 1, \dots$, form a complete orthogonal system in $L^2[-1, 1]$. We refer this to [17, p. 25]. By the orthogonality of the spherical harmonics, the functions in the form

$$\begin{cases} v_{l,m}(x; k) := a_{l,m} j_l(kr) Y_l^m(\hat{x}); \\ w_{l,m}(x; k) := \frac{b_{l,m} y_l(r; k)}{r} Y_l^m(\hat{x}) \end{cases} \quad (1.6)$$

satisfy the first two equations in (1.1) independently [17, p. 227] for $r \geq R_0$.

Now we consider $k \in \mathbb{C}$, $a_{l,m}$, and $b_{l,m}$, if any, satisfies the following independent system for $l \in \mathbb{N}_0$ and $-l \leq m \leq l$,

$$\begin{cases} a_{l,m} j_l(kr)|_{r=R_0} = b_{l,m} \frac{y_l(r; k)}{r}|_{r=R_0}; \\ a_{l,m} [j_l(kr)]'|_{r=R_0} = b_{l,m} \left[\frac{y_l(r; k)}{r} \right]'|_{r=R_0}. \end{cases} \quad (1.7)$$

In terms of elementary linear algebra, the existence of the coefficients $a_{l,m}$ and $b_{l,m}$ are equivalent to finding the zeros of the following functional determinant:

$$D_l(k; R_0) := \det \begin{pmatrix} j_l(kr)|_{r=R_0} & -\frac{y_l(r; k)}{r}|_{r=R_0} \\ \{j_l(kr)\}'|_{r=R_0} & -\left\{ \frac{y_l(r; k)}{r} \right\}'|_{r=R_0} \end{pmatrix} \quad (1.8)$$

$$= -j_l(kR_0) \frac{y_l'(R_0; k)}{R_0} + j_l(kR_0) \frac{y_l(R_0; k)}{R_0^2} + k j_l'(R_0) \frac{y_l(R_0; k)}{R_0}, \quad (1.9)$$

in which the system is independent of m and \hat{x} . The forward problem describes the distribution of the zeros of $D_l(k; R_0)$, while the inverse problem specifies the index of refraction n by the zero set. In [11, 12, 13, 28], we have discussed the methods to find the zeros of $D_l(k; R_0)$.

Let $k \in \mathbb{C}$ be a possible eigenvalue of (1.7). Applying the analytic continuation of the Helmholtz equation and Rellich's lemma [17, p. 32, 33; p. 222], the solutions parameterized by k solve

$$\begin{cases} w(x; k) \equiv v(x; k), & x \notin D; \\ \frac{\partial w(x; k)}{\partial \nu} \equiv \frac{\partial v(x; k)}{\partial \nu}, & x \notin D, \end{cases} \quad (1.10)$$

in the simple domain D . Conversely, (1.10) implies (1.8) by the uniqueness of the system (1.6) at $r = R_0$.

We note that the representation (1.2) and (1.3) can merely hold outside $|x| \geq R_0$, so we seek to extend the representation into $|x| \leq R_0$. Let $\hat{x}_1 \in \mathbb{S}^2$ be a given incident direction satisfying the following geometric condition:

$$\text{the line segment from } (R_0, \hat{x}_1) \text{ to } (r_1, \hat{x}_1), r_1 \leq R_0, \text{ lies outside } D \text{ with } (r_1, \hat{x}_1) \in \partial D. \quad (1.11)$$

For this \hat{x}_1 , we extend each Fourier coefficient $y_l(r; k)$ with $k \in \mathbb{C}$ determined by system (1.7) for all l , toward the origin until it meets the boundary ∂D at (r_1, \hat{x}_1) . Along the given \hat{x}_1 , we apply the differential operator $\Delta + k^2 n$ with

$$\Delta = \frac{1}{r^2} \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2 \sin \varphi} \frac{\partial}{\partial \varphi} \sin \varphi \frac{\partial}{\partial \varphi} + \frac{1}{r^2 \sin^2 \varphi} \frac{\partial^2}{\partial \theta^2} \quad (1.12)$$

to $\{w_{l,m}(x)\}$, which accordingly can solve the problem (1.1) with the manmade index of refraction $n(x) = n(r\hat{x}) = n(r\hat{x}_1)$ for all $\hat{x} \in \mathbb{S}^2$. More importantly, the analytic continuation (1.10) and the interior transmission condition imply the following ODE:

$$\begin{cases} y_l''(r; k) + (k^2 n(r\hat{x}_1) - \frac{l(l+1)}{r^2}) y_l(r; k) = 0; \\ D_l(k; r_1) = 0. \end{cases} \quad (1.13)$$

If there is merely one intersection point for $[0, R_0] \times \hat{x}_1$ with ∂D , then we set the initial conditions to (1.13)

$$y_0(0; k) = 0, y_0'(0; k) = 1; y_l(0; k) = 0, y_l'(0; k) = 0, l \in \mathbb{N}. \quad (1.14)$$

That is,

$$D_l(k; 0) = 0. \quad (1.15)$$

We refer to the details to [1, 17, 28]. For $l \geq 0$, we can take $a_{l,m} = b_{l,m} = 1$ in (1.7) by the uniqueness implied by (1.14). Now, the uniqueness of the ODE (1.7) is valid up to the boundary ∂D ,

$$\begin{cases} a_{l,m} j_l(kr) = b_{l,m} \frac{y_l(r; k)}{r}; \\ a_{l,m} [j_l(kr)]' = b_{l,m} [\frac{y_l(r; k)}{r}]', r = r_1, \hat{x}_1 \in \mathbb{S}^2. \end{cases} \quad (1.16)$$

That is, $D_l(k; r_1) = D_l(k; R_0)$ by the uniqueness of ODE (1.13) along the line segment (R_0, \hat{x}_1) to (r_1, \hat{x}_1) .

In general, the solution $y_l(r)$ depends on the incident direction \hat{x} whenever entering the perturbation, so we denote the extended solution of (1.13) as $\hat{y}_l(r; k)$, and the functional determinant as $\hat{D}_l(k; r_1)$. Thus, (1.13) is rephrased as

$$\begin{cases} \hat{y}_l''(r; k) + (k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2}) \hat{y}_l(r; k) = 0; \\ \hat{D}_l(k; 0) = 0, \hat{D}_l(k; r_1) = 0. \end{cases} \quad (1.17)$$

The eigenvalues of (1.17) are discussed in [1, 11, 12, 13] by the singular Sturm-Liouville theory in [6, 7, 8].

However, the domain D is not starlike in general. Instead of (1.16), we now ask for any $k \in \mathbb{C}$ such that, the following system holds

$$\begin{cases} \hat{a}_{l,m} j_l(kr)|_{r=\hat{r}} = \hat{b}_{l,m} \frac{\hat{y}_l(r; k)}{r}|_{r=\hat{r}}; \\ \hat{a}_{l,m} [j_l(kr)]'|_{r=\hat{r}} = \hat{b}_{l,m} [\frac{\hat{y}_l(r; k)}{r}]'|_{r=\hat{r}}, \hat{r} \in \hat{R}, \end{cases} \quad (1.18)$$

in which \hat{R} is the intersection set along the incident angle \hat{x} defined by

$$\hat{R} := \{\hat{r} | (\hat{r}, \hat{x}) \in \partial D\} = \{\hat{r}_1, \dots, \hat{r}_M\}, \quad (1.19)$$

and $\hat{y}_l(r; k)$ are defined by the solutions of first line of equation (1.13) with initial condition (1.18).

In general, we assume \hat{R} to be a finite discrete set and $\hat{r}_1 < \hat{r}_2 < \dots < \hat{r}_{\hat{M}}$. In the case that (α, \hat{x}) , (β, \hat{x}) , and (γ, \hat{x}) are any three consecutive points along the incident direction \hat{x} . Whenever (β, \hat{x}) is a tangent point at the boundary, we disregard it and consider the line segment from (α, \hat{x}) to (γ, \hat{x}) as either completely inside or outside the perturbation. Without loss of generality, we assume that \hat{R} contains no tangent point. See Fig. ??.

Firstly starting with the first segment into the perturbation, we discuss the well-posedness of the initial value problem starting at $\hat{r}_{\hat{M}}$:

$$\begin{cases} \hat{y}_l''(r) + (k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2})\hat{y}_l(r) = 0, \hat{r}_{\hat{M}-1} \leq r \leq \hat{r}_{\hat{M}}; \\ \hat{D}_l(k; \hat{r}_{\hat{M}}) = 0, \end{cases} \quad (1.20)$$

which has an unique solution inward to $\hat{r}_{\hat{M}-1}$ given $j_l(k\hat{r})$ as a known function. The behavior of the solution is understood by the singular Sturm-Liouville theory in Section 2. Because $\hat{r}_{\hat{M}}$ is the first intersection, $w_{l,m}(\hat{x}\hat{r}_{\hat{M}}; k) = v_{l,m}(\hat{x}\hat{r}_{\hat{M}}; k)$, the uniqueness of (1.20) holds to the $\hat{r}_{\hat{M}-1}$ for $k \in \mathbb{C}$. Because of (1.10) and the construction of (1.6), the unique analytic continuation of $w_{l,m}(x, k)$ holds outside D as well as along the \hat{x} inside D . Thus, $\hat{D}_l(k; \hat{r}_{\hat{M}-1}) = 0$ holds by analytic continuation of the Helmholtz equation. More importantly, it filters out a discrete set of eigenvalues of

$$\begin{cases} \hat{y}_l''(r) + [k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2}]\hat{y}_l(r) = 0, \hat{r}_{\hat{M}-1} \leq r \leq \hat{r}_{\hat{M}}; \\ \hat{D}_l(k; \hat{r}_{\hat{M}}) = 0, \hat{D}_l(k; \hat{r}_{\hat{M}-1}) = 0. \end{cases} \quad (1.21)$$

Let k^T be one of its eigenvalues. Leaving the perturbation at $\hat{r}_{\hat{M}-1}$, the k^T defines another ODE system:

$$\begin{cases} \hat{y}_l''(r) + [k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2}]\hat{y}_l(r) = 0, \hat{r}_{\hat{M}-2} \leq r \leq \hat{r}_{\hat{M}-1}; \\ \hat{D}_l(k; \hat{r}_{\hat{M}-1}) = 0, \hat{D}_l(k; \hat{r}_{\hat{M}-2}) = 0, \end{cases} \quad (1.22)$$

in which $\hat{D}_l(k^T; \hat{r}_{\hat{M}-2}) = 0$ holds due to the analytic continuation of Helmholtz equation. The same k^T appears at $\hat{r}_{\hat{M}-2}$ and ready to define another new ODE.

$$\begin{cases} \hat{y}_l''(r) + [k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2}]\hat{y}_l(r) = 0, \hat{r}_{\hat{M}-3} \leq r \leq \hat{r}_{\hat{M}-2}; \\ \hat{D}_l(k; \hat{r}_{\hat{M}-2}) = 0, \hat{D}_l(k; \hat{r}_{\hat{M}-3}) = 0. \end{cases} \quad (1.23)$$

By analytic continuation, k^T satisfies (1.21), (1.22), and (1.23), and appears at $\hat{r}_{\hat{M}-3}$ as well. More importantly, the system (1.22) produces new eigenvalues that appear at $\hat{r}_{\hat{M}-3}$ and consecutively into each intersection interval. Thus, we have

$$\begin{cases} \hat{y}_l''(r) + [k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2}]\hat{y}_l(r) = 0, \hat{r}_0 := 0 < r < \infty; \\ \hat{D}_l(k; \hat{r}_0) = 0, \hat{D}_l(k; \hat{r}_1) = 0, \dots, \hat{D}_l(k; \hat{r}_{\hat{M}}) = 0. \end{cases} \quad (1.24)$$

Each element of the zero set of $\hat{D}_l(k; \hat{r}_j)$ defines an initial value problem in the neighboring interval. There is an uniqueness to the solution of (1.24) by the piecewise construction as shown above, and we call the extended solution $\hat{y}(r; k)$ for each k the eigenvalue tunneling in interior transmission problem. Such a construction can be set to initiate at \hat{r}_0 and tunnels to the infinity. We have already discussed the simple case as in the starlike domains [11, 12, 13]:

$$\begin{cases} \hat{y}_l''(r) + (k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2})\hat{y}_l(r) = 0, \hat{r}_0 < r < \infty; \\ \hat{D}_l(k; \hat{r}_0) = 0, \hat{D}_l(k; \hat{r}_1) = 0. \end{cases} \quad (1.25)$$

With the initial condition $\hat{D}_l(k; 0) = 0$, the function $\hat{D}_l(k; \hat{r}_1)$ is an entire function of exponential type [6, 7, 8, 11, 12, 13, 30]. The eigenvalues are its zeros. Without loss of generality, we take \hat{r}_0 as the reference point. Thus, the eigenvalues of (1.25) form a discrete set in \mathbb{C} , accumulate into the eigenvalues of (1.24), and tunnel to the infinity.

Conversely, once we find an eigenvalue of (1.24) for some l along some \hat{x} , it solves (1.7) by the uniqueness of ODE and then (1.10) by the analytic continuation of Helmholtz equation. Whenever we collect all such eigenvalues from each incident $\hat{x} \in \mathbb{S}^2$, they are interior transmission eigenvalues of (1.1) by analytic continuation. The geometric characteristics of the perturbation are connected by rays of ODE system to the far fields.

Theorem 1.1. *Let n^1, n^2 be two unknown indices of refraction as assumed in (1.1). If they have the same set of eigenvalues, then $n^1 \equiv n^2$.*

2 Asymptotic Expansions and Cartwright-Levinson Theory

To study the functional determinants $\hat{D}_l(k; \hat{r})$, we collect the following asymptotic behaviors of $\hat{y}_l(r; k)$ and $\hat{y}'_l(r; k)$. For $l = 0$, we apply the Liouville transformation [6, 7, 8, 16, 17, 30]:

$$\hat{z}_0(\hat{\xi}) := [n(r\hat{x})]^{\frac{1}{4}} \hat{y}_0(r; k), \quad (2.1)$$

where

$$\hat{B}(r) := \hat{\xi}(r) = \int_0^r [n(\rho\hat{x})]^{\frac{1}{2}} d\rho, \quad 0 \leq r \leq \hat{r}_1. \quad (2.2)$$

Here we recall that n is 1 outside D . For simplicity of the notation, we drop all the superscripts about \hat{x} whenever the context is clear.

Definition 2.1. Let $f(z)$ be an integral function of order ρ , and let $N(f, \alpha, \beta, r)$ denote the number of the zeros of $f(z)$ inside the angle $[\alpha, \beta]$ and $|z| \leq r$. We define the density function as

$$\Delta_f(\alpha, \beta) := \lim_{r \rightarrow \infty} \frac{N(f, \alpha, \beta, r)}{r^\rho}, \quad (2.3)$$

and

$$\Delta_f(\beta) := \Delta_f(\alpha_0, \beta), \quad (2.4)$$

with some fixed $\alpha_0 \notin E$ such that E is at most a countable set.

Lemma 2.2. *The functional determinant $\hat{D}_l(k; r)$ is of order one and of type $r + \hat{B}(r)$, $\hat{r}_0 \leq r \leq \hat{r}_1$. In particular,*

$$\Delta_{\hat{D}_l(k; r)}(-\epsilon, \epsilon) = \frac{r + \hat{B}(r)}{\pi}. \quad (2.5)$$

Proof. We begin with (1.8).

$$\begin{aligned} \hat{D}_l(k; r) &= -j_l(kr) \frac{\hat{y}'_l(r; k)}{r} + j_l(kr) \frac{\hat{y}_l(r; k)}{r^2} + k j'_l(kr) \frac{\hat{y}_l(r; k)}{r} \\ &= \frac{k j'_l(kr) \hat{y}_l(r; k)}{r} \left\{ 1 - \frac{1}{k} \frac{j_l(kr)}{j'_l(kr)} \frac{\hat{y}'_l(r; k)}{\hat{y}_l(r; k)} + \frac{1}{kr} \frac{j_l(kr)}{j'_l(kr)} \right\} \\ &= \frac{k j'_l(kr) \hat{y}_l(r; k)}{r} \left\{ \hat{\alpha}_l(k) + O\left(\frac{1}{k}\right) \right\}, \end{aligned} \quad (2.6)$$

in which

$$\hat{\alpha}_l(k) := 1 - \frac{1}{k} \frac{j_l(kr)}{j'_l(kr)} \frac{\hat{y}'_l(r; k)}{\hat{y}_l(r; k)}; \quad (2.7)$$

We have $\frac{j_l(k\hat{r})}{j'_l(kr)} = O(1)$ outside the zeros of $j'_l(kr)$; similarly, $\frac{\hat{y}'_l(r; k)}{\hat{y}_l(r; k)} = O(k)$ outside the zeros of $\hat{y}_l(r; k)$. The term $\hat{\alpha}(k)$ is bounded and bounded away from zero outside the zeros of $j'_l(kr)$ and $\hat{y}_l(r; k)$ on real axis.

Consequently, we can compute the Lindelöf's indicator function [11, 12, 13, 25, 26] for $\hat{D}_l(k)$ for (2.6):

$$h_{\hat{D}_l(k)}(\theta) = h_{j'_l(k\hat{r})}(\theta) + h_{\hat{y}_l(\hat{r}; k)}(\theta) = (\hat{r} + \hat{B}(\hat{r})) |\sin \theta|, \quad \theta \in [0, 2\pi], \quad \hat{r}_0 \leq \hat{r} \leq \hat{r}_1, \quad (2.8)$$

in which $\hat{B}(\hat{r}) = \int_0^{\hat{r}} [n(\rho\hat{x})]^{\frac{1}{2}} d\rho$, and we apply an inequality of Lindelöf's indicator function in [25, p. 51].

In case that $\hat{\alpha}_l(k) \equiv 0$, instead of (2.8), we have

$$h_{\hat{D}_l(k)}(\theta) = h_{j_l(k\hat{r})}(\theta) + h_{\hat{y}_l(\hat{r};k)}(\theta) = (\hat{r} + \hat{B}(\hat{r}))|\sin \theta|, \theta \in [0, 2\pi], \hat{r}_0 \leq \hat{r} \leq \hat{r}_1, \quad (2.9)$$

Referring Cartwright theory to [25, p.251], we conclude that $\hat{D}_l(k)$ is of Cartwright's class, and the lemma is thus proven. \square

Lemma 2.3. *The meromorphic function $\hat{\alpha}_l(k) \equiv 0$ if and only if $n(r\hat{x}) \equiv 1$ along the incident angle \hat{x} .*

Proof. We begin with $\frac{\hat{y}_l'(\hat{r};k)}{\hat{y}_l(\hat{r};k)} \equiv k \frac{j_l'(k\hat{r})}{j_l(k\hat{r})}$ as a meromorphic function in k . Referring to [6, 7, 8, 30], $j_l'(k\hat{r})$ has zeros asymptotically distributed near the zeros of $\cos(k\hat{r})$; $j_l(k\hat{r})$ near the zero set of $\sin(k\hat{r})$. A similar property holds for $\hat{y}_l'(\hat{r};k)$ and $\hat{y}_l(\hat{r};k)$. Hence, whenever $j_l'(k\hat{r})$ has a zero, $\hat{y}_l'(\hat{r};k)$ has a zero; $\hat{y}_l(\hat{r};k)$ has one whenever $j_l(k\hat{r})$ has. The two perturbations, $j_l(k\hat{r})$ and $\hat{y}_l(\hat{r};k)$, have the same set of Neumann and Dirichlet eigenvalue. By the inverse spectral uniqueness of Sturm-Liouville problem [1, 30], we have $n \equiv 1$. The sufficient condition is obvious. This proves the lemma. \square

Thus, Lemma 2.2 merely describes the eigenvalue density of the problem (1.25). To describe the density for (1.24), we may apply the translation invariant properties of interior transmission eigenvalues [14, Lemma 1.3]. Alternatively, we may consider from the point of view of uniqueness theorem of ODE as in the Introduction: In $[\hat{r}_1, \hat{r}_2]$, except the previous eigenvalues of (1.25), we consider the new eigenvalue density of the problem from the second interval, that is,

$$\begin{cases} \hat{y}_l''(r; k) + (k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2})\hat{y}_l(r; k) = 0, \hat{r}_1 < r < \hat{r}_2; \\ \hat{D}_l(k; \hat{r}_1) = 0, \hat{D}_l(k; \hat{r}_2) = 0, \end{cases} \quad (2.10)$$

is

$$0, \quad (2.11)$$

because $\frac{\hat{y}_l(r; k)}{r}$ and $j_l(rk)$ satisfy the same differential equation and initial condition at \hat{r}_1 until \hat{r}_2 . Thus, there is only trivial eigenfunctions in $[\hat{r}_1, \hat{r}_2]$. One can prove inductively the density in each intersection interval, so we state the following theorem.

Theorem 2.4. *Let $\hat{N}(\alpha, \beta, R)$ denote the number of the eigenvalues of (1.24) inside the angle $[\alpha, \beta]$ and $|z| \leq R$ in \mathbb{C} . We define the density function as*

$$\hat{\Delta}(\alpha, \beta) := \lim_{R \rightarrow \infty} \frac{\hat{N}(\alpha, \beta, R)}{R}. \quad (2.12)$$

Then

$$\hat{\Delta}(-\epsilon, \epsilon) = \frac{|\hat{r}\chi_D| + \int_0^\infty n^{\frac{1}{2}}(\rho\hat{x})\chi_D d\rho}{\pi}, \quad (2.13)$$

in which χ_D is 1 in D ; zero otherwise, and $|\cdot|$ is the Lebesgue measure.

We elaborate on the eigenvalue tunneling.

Proposition 2.5. *If k satisfies the following ODE for some $l \geq 0$:*

$$\begin{cases} \hat{y}_l''(r) + (k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2})\hat{y}_l(r) = 0, \hat{r}_0 := 0 < r < \infty; \\ \hat{D}_l(k; \hat{r}_0) = 0; \\ \hat{D}_l(k; \hat{r}_1) = 0, \end{cases} \quad (2.14)$$

then the k is the eigenvalue to the following system for the same l as well:

$$\begin{cases} \hat{y}_l''(r) + (k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2})\hat{y}_l(r) = 0, 0 < r < \infty; \\ \hat{D}_l(k; \hat{r}_0) = 0, \hat{D}_l(k; \hat{r}_1) = 0, \dots, \hat{D}_l(k; \hat{r}_M) = 0. \end{cases} \quad (2.15)$$

Proof. Let the eigenvalue k solve the system of (2.14) for some $l \geq 0$. The first two equations there give an entire function in k , and the third condition implies that the eigenvalues k form a discrete set in \mathbb{C} [13]. For this k given by (2.14), we consider the ODE system with the mixed boundary condition:

$$\begin{cases} \hat{y}_l''(r) + (k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2})\hat{y}_l(r) = 0, \hat{r}_1 < r < \hat{r}_2; \\ \hat{D}_l(k; \hat{r}_1) = 0; \\ \hat{D}_l(k; \hat{r}_2) = 0. \end{cases} \quad (2.16)$$

The second equation above with $\hat{D}_l(k; \hat{r}_1) = 0$ gives a new initial value problem at \hat{r}_1 , that is,

$$\begin{cases} j_l(kr)|_{r=\hat{r}_1} = \frac{\hat{y}_l(r; k)}{r}|_{r=\hat{r}_1}; \\ [j_l(kr)]'|_{r=\hat{r}_1} = [\frac{\hat{y}_l(r; k)}{r}]'|_{r=\hat{r}_1}. \end{cases} \quad (2.17)$$

Similarly, $\hat{y}_l(\hat{r}; k)$ initiates at \hat{r}_2 again and then consecutively into the infinity with the same k . Moreover, any eigenvalue k of (2.14) satisfies $\hat{D}_l(k; \hat{r}_2) = 0$ in (2.16) by the unique continuation result (1.10) of $w_{l,m}(x; k)$ and similarly at each intersection points in \hat{R} . Hence, the given k satisfies

$$\hat{D}_l(k; \hat{r}_0) = 0, \hat{D}_l(k; \hat{r}_1) = 0, \dots, \hat{D}_l(k; \hat{r}_{\hat{M}}) = 0. \quad (2.18)$$

This proves the lemma. \square

In general, we can take \hat{r}_j as the reference point.

Proposition 2.6. *If k satisfies the following ODE for some $l \geq 0$:*

$$\begin{cases} \hat{y}_l''(r) + (k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2})\hat{y}_l(r) = 0, \hat{r}_j < r < \hat{r}_{j+1}; \\ \hat{D}_l(k; \hat{r}_j) = 0; \\ \hat{D}_l(k; \hat{r}_{j+1}) = 0, \end{cases} \quad (2.19)$$

then the k is the eigenvalue to the following system

$$\begin{cases} \hat{y}_l''(r) + (k^2 n(r\hat{x}) - \frac{l(l+1)}{r^2})\hat{y}_l(r) = 0, \hat{r}_j < r < \infty; \\ \hat{D}_l(k; \hat{r}_j) = 0, \hat{D}_l(k; \hat{r}_{j+1}) = 0, \dots, \hat{D}_l(k; \hat{r}_{\hat{M}}) = 0. \end{cases} \quad (2.20)$$

3 A Proof of Theorem 1.2

Proof. Let k be an eigenvalue of (1.1). Thus, k is an eigenvalue of (1.7) for some l , and then extends toward the origin along some $\hat{x} \in \mathbb{S}^2$ for any l . Therefore, it is an eigenvalue of (2.15) for some l along some $\hat{x} \in \mathbb{S}^2$. Conversely, if k is an eigenvalues of (2.15), then it solves (1.7) at $|x| = R_0$. The system (1.7) is radially symmetrically defined. Thus, with this same k , we can solve the ODE (2.15) along any other direction. Thus the Fourier coefficients $\{\hat{y}_l(r; k)\}$ is defined for all angle \hat{x} . Accordingly, $w(x; k)$ is extended in \mathbb{R}^3 from the far fields with the given k .

If n^i , $i = 1, 2$, are two indices of refraction with identical interior transmission eigenvalues, then the spectrum satisfies $\hat{D}_l(k; \hat{r}) = 0$ for all \hat{r} and some l for any \hat{x} . This is an one dimensional inverse Sturm-Liouville problem [1, 11, 12, 13]. In particular, we start with $j = 0$. That is, (2.14) holds for one l along some $\hat{x} \in \mathbb{S}^2$. Hence, inverse uniqueness of the Bessel operator [7, Theorem 1.2, Theorem 1.3] proves that $n^1(r\hat{x}) \equiv n^2(r\hat{x})$ in $[\hat{r}_0, \hat{r}_1]$. In this case, we have $n^1(r\hat{x}) = n^2(r\hat{x})$ for $r \in [\hat{r}_0, \hat{r}_1]$. The argument holds for all j inductively. This proves Theorem 1.1. \square

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